General Relativity

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[?] 重力波

時空に摂動が加わり、計量テンソルが

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & & & \mathbf{0} \\ & 1+E_+ & & \\ & & 1-E_+ & \\ \mathbf{0} & & & 1 \end{pmatrix},\ (g^{\mu\nu}) = \begin{pmatrix} -1 & & & \mathbf{0} \\ & 1 & & \\ & & 1 & \\ \mathbf{0} & & & 1 \end{pmatrix} \tag{1}$$

のように変化したとする。ここで $E_+ = \hat{E}_+ e^{i(\omega t - kz)}$ のように z 方向に伝わる波動の形をしているとする。

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}) \tag{2}$$

より

$$\Gamma^0_{0\beta} = \frac{1}{2}g^{00}(g_{00,\beta} + g_{0\beta,0} - g_{0\beta,0}) = 0,\ \Gamma^0_{11} = \frac{1}{2}g^{00}(g_{01,1} + g_{01,1} - g_{11,0}) = -\frac{1}{2}g^{00}g_{11,0} = \frac{1}{2}E_{+,0},\ \Gamma^0_{11} = -\frac{1}{2}g^{00}g_{22,0} = -\frac{1}{2}E_{+,0}$$

$$\Gamma^0_{33} = 0,\ \Gamma^0_{ij} = 0\ (i \neq j),\ \Gamma^1_{00} = \frac{1}{2}g^{11}(g_{10,0} + g_{10,0} - g_{00,1}) = 0,\ \Gamma^1_{10} = \frac{1}{2}g^{11}(g_{11,0} + g_{10,1} - g_{10,1}) = \frac{1}{2}g^{11}g_{11,0} = \frac{1}{2}E_{+,0}$$

$$\Gamma^1_{11} = \Gamma^1_{12} = 0,\ \Gamma^1_{13} = \frac{1}{2}g^{11}g_{11,3} = \frac{1}{2}E_{+,3},\ \Gamma^1_{20} = \Gamma^1_{22} = \Gamma^1_{23} = 0,\ \Gamma^1_{30} = \Gamma^1_{33} = 0,\ \Gamma^2_{00} = \Gamma^2_{10} = \Gamma^2_{30} = \Gamma^2_{11} = \Gamma^2_{12}\Gamma^2_{13} = 0$$

$$\Gamma^2_{20} = \frac{1}{2}g^{22}g_{22,0} = -\frac{1}{2}E_{+,0},\ \Gamma^2_{22} = 0,\ \Gamma^2_{23} = \frac{1}{2}g^{22}g_{22,3} = -\frac{1}{2}E_{+,3},\ \Gamma^3_{00} = \Gamma^3_{10} = \Gamma^3_{20} = \Gamma^3_{30} = \Gamma^3_{3i} = 0,\ \Gamma^3_{11} = -\frac{1}{2}g^{33}g_{11,3} = -\frac{1}{2}E_{+,3}$$

$$\Gamma^3_{12} = \Gamma^3_{13} = 0,\ \Gamma^3_{22} = -\frac{1}{2}g^{33}g_{22,3} = \frac{1}{2}E_{+,3},\ \Gamma^3_{23} = \Gamma^3_{33} = 0$$

よって

$$R_{00} = \Gamma^{\mu}_{00,\mu} - \Gamma^{\mu}_{0\mu,0} = -\frac{1}{2}E_{+,00} + \frac{1}{2}E_{+,00} = 0 \tag{3}$$

$$R_{11} = \Gamma^{\mu}_{11,\mu} - \Gamma^{\mu}_{1\mu,1} = \frac{1}{2}E_{+,00} - \frac{1}{2}E_{+,33} \tag{4}$$

$$R_{22} = \Gamma^{\mu}_{22,\mu} - \Gamma^{\mu}_{2\mu,2} = -\frac{1}{2}E_{+,00} + \frac{1}{2}E_{+,33} \tag{5}$$

$$R_{33} = \Gamma^{\mu}_{33,\mu} - \Gamma^{\mu}_{3\mu,3} = -\frac{1}{2}E_{+,33} + \frac{1}{2}E_{+,33} = 0 \tag{6}$$

$$R = g^{\mu\nu}R_{\mu\nu} = R^{\mu}{}_{\mu} = 0 \tag{7}$$

これより、アンシュタインテンソルは

$$G_{11} = R_{11} - \frac{1}{2}g_{11}R = \frac{1}{2}E_{+,00} - \frac{1}{2}E_{+,33} \tag{8}$$

今、物質は存在しない真空中を考えると $T^{\mu\nu} = 0$ より

$$(\text{Einstein eq.}) \implies \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)E_+ = 0 \tag{9}$$

これは光速度 c で伝搬する波を表す波動方程式である。真空中を c で伝搬する時空の摂動、これを重力波と呼ぶ。