General Relativity

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[?] ビアンキの恒等式

$$[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = \nabla_\lambda(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu) - (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\nabla_\lambda \tag{1}$$

$$[\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] = \nabla_\mu(\nabla_\nu\nabla_\lambda - \nabla_\lambda\nabla_\nu) - (\nabla_\nu\nabla_\lambda - \nabla_\lambda\nabla_\nu)\nabla_\mu \tag{2}$$

$$[\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] = \nabla_\nu(\nabla_\lambda\nabla_\mu - \nabla_\mu\nabla_\lambda) - (\nabla_\lambda\nabla_\mu - \nabla_\mu\nabla_\lambda)\nabla_\nu \tag{3}$$

より

$$\{[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]]\}A^\alpha = 0 \tag{4}$$

これと

$$\begin{aligned}[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]]A^\alpha &= \nabla_\lambda[\nabla_\mu, \nabla_\nu]A^\alpha - [\nabla_\mu, \nabla_\nu](\nabla_\lambda A^\alpha) = \nabla_\lambda(R^\alpha_{\gamma\mu\nu}A^\gamma) - (-R^\gamma_{\lambda\mu\nu}\nabla_\gamma A^\alpha + R^\alpha_{\gamma\mu\nu}\nabla_\lambda A^\gamma) \\ &= (\nabla_\lambda R^\alpha_{\gamma\mu\nu})A^\gamma + R^\gamma_{\lambda\mu\nu}\nabla_\gamma A^\alpha \end{aligned} \tag{5}$$

より

$$(\nabla_\lambda R^\alpha_{\gamma\mu\nu} + \nabla_\mu R^\alpha_{\gamma\nu\lambda} + \nabla_\nu R^\alpha_{\gamma\lambda\mu})A^\gamma + (R^\gamma_{\lambda\mu\nu} + R^\gamma_{\mu\nu\lambda} + R^\gamma_{\nu\lambda\mu})\nabla_\gamma A^\alpha = 0 \tag{6}$$

$A^\gamma, \nabla_\gamma A^\alpha$ は任意のベクトルなので、これが恒等的に成り立つためにはこの 2 つの係数は 0 でなければならない。

$$\therefore \quad \nabla_\lambda R^\alpha_{\gamma\mu\nu} + \nabla_\mu R^\alpha_{\gamma\nu\lambda} + \nabla_\nu R^\alpha_{\gamma\lambda\mu} = 0 \tag{7}$$

$$R^\gamma_{\lambda\mu\nu} + R^\gamma_{\mu\nu\lambda} + R^\gamma_{\nu\lambda\mu} = 0 \tag{8}$$

これをビアンキの恒等式という。